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Note on formulas for the drag of a sphere

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Standard approximations expressing the drag of a sphere as a function of Reynolds number are reappraised in the light of the evident requirement that drag reverses with the direction of motion. It is thereby highlighted that the relation between the drag and the velocity of a sphere is not analytic. Another, simpler example is cited to illustrate a non-analytic relation between physical properties, which is appreciated to be a common feature of hydrodynamic models that rely on the abstract notion of an infinite incompressible fluid.

1. Preamble

This note aims to clarify an issue about which I have for many years been in mild and friendly contention with Professor Milton Van Dyke. The issue is superficially trivial, but it depends on points of interpretation that are delicate enough to deserve the present detailed commentary. The following discussion serves at least to ventilate a curious attribute of a classic problem in hydrodynamics, which aspect seems to have received little attention previously although it was implicitly covered in an investigation by Chester (1962).

According to the celebrated method of approximation devised by Oseen (1910) and to improvements upon it due to Proudman & Pearson (1957) and many others, the drag D experienced by a solid sphere of radius a moving with velocity U in an infinite incompressible fluid of density ρ and viscosity $\mu = \rho\nu$ is given by

$$D = 6\pi\mu a U[1+\tfrac{3}{8}R+\tfrac{9}{40}R^2\ln R+O(R^2)] \tag{1}$$

(cf. Van Dyke 1975, pp. 5, 234). Here $R = aU/\nu$ is the Reynolds number, and all derivations of this asymptotic expansion for small R are based on the presumption that U is a positive quantity, which makes the definition of R unequivocal. In terms of a dimensionless drag coefficient $C_D = D/\rho a^2U^2$, the result (1) can be rewritten

$$C_D = \frac{6\pi}{R}[1+\tfrac{3}{8}R+\tfrac{9}{40}R^2\ln R+O(R^2)]; \tag{2}$$

but this formulation tends to disguise the aspect to be discussed as follows. (Approximations to higher order in R have been worked out by many contributors to the subject, but for present purposes it is needless to cite them.)

The Stokes approximation $D = 6\pi\mu a U$ ignoring inertial effects in the fluid plainly reflects the group invariance of the idealized hydrodynamic problem. Strictly speaking, D so expressed is not the drag on a sphere moving with velocity U but is rather the external force needed to drive the motion. Equivalently, in keeping with the standard mathematical model used for derivations, D is the drag on a stationary

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sphere in a fluid that is assigned a uniform velocity U at infinity. In either interpretation, D evidently has the same direction as U; and because the hydrodynamic problem has the symmetry of the whole rotation group in $\mathbb{R}^3$, the Stokes result can at once be generalized to a vector formulation $\boldsymbol{D} = 6\pi\mu a\boldsymbol{U}$. In particular, it admits the discrete Z_2-symmetry $U \to -U$, $D \to -D$.

The complete hydrodynamic problem including inertial effects has the same symmetry. Therefore, if the definition of R is extended to negative as well as positive and zero values of U, the expression (1) appears to remain consistent only if the factor in brackets on the right-hand side is the even function of R given by writing $|R|$ in place of R. Thus one may provisionally infer

$$D = 6\pi\mu aU[1+\tfrac{3}{8}|R|+\tfrac{9}{40}R^2\ln|R|+\dots] \tag{3}$$

to be the correct generalization of (1) covering the requisite symmetry. This simple observation is the main content of this note. It is significant in showing that the infinite-fluid model predicts D to be a *non-analytic* function of U, having a singularity at $R = 0$ even stronger than is suggested by the logarithmic term in (1). Implications of this curious feature will be discussed later, together with a plausible conjecture about its absence from any hydrodynamic model in which the fluid is bounded.

One may reasonably argue, as Professor Van Dyke does, that the Reynolds number is introduced in flow problems merely as a scaling parameter, so that the attribution of sign to it should be irrelevant to analyses such as the usual derivation of (1). As will be confirmed in §2, however, the appearance of R in non-dimensional forms of the Navier–Stokes equations is in fact tied to a sign implication, the recognition of which justifies (3) rigorously as the proper generalization of (1). It will be of interest to see precisely how the non-analytic function $|R|$ of the parameter R arises in the analysis. Even without such a check, the outcome is wholly to be expected. For instance, the correction of D found originally by Oseen and represented by the second term on the right-hand side of (1) is seen to be just $\frac{9}{4}\pi\rho a^2U^2$ when factors μ in R and the Stokes drag are cancelled; and according to (3) it is generalized to $\frac{9}{4}\pi\rho a^2U^2\,\mathrm{sgn}(U)$, as evidently required for the correction to be more generally applicable.

2. The mathematical problem

The problem is commonly posed as one of steady axisymmetric motion about a fixed sphere; the fluid is assigned a uniform velocity U at infinity. In terms of spherical polar coordinates (r,θ,ϕ), whose axis is aligned with the flow at infinity, the motion is independent of the azimuthal angle ϕ; and it is representable by a Stokes stream function $\psi(r,\theta)$ such that the components of velocity in the directions of r and θ increasing are respectively $u = (r^2\sin\theta)^{-1}\psi_\theta$ and $v = -(r\sin\theta)^{-1}\psi_r$. The equation of mass conservation $(\nabla\cdot(u,v,0) = 0)$ is thus automatically satisfied; and the elimination of pressure between the Navier–Stokes equations leads to a semilinear fourth-order equation for ψ (in effect the azimuthal vorticity equation) which in dimensional form is

$$\nu\mathrm{L}^2\psi = \frac{1}{r^2\sin\theta}\left(\psi_\theta\frac{\partial}{\partial r}-\psi_r\frac{\partial}{\partial\theta}+2\psi_r\cot\theta-\frac{2}{r}\psi_\theta\right)\mathrm{L}\psi, \tag{4}$$

with
$$\mathrm{L} = \frac{\partial^2}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$$

(Goldstein 1938, p. 115). The problem is completed by the boundary conditions of vanishing velocity at the surface $r = a$ of the sphere, namely

$$\psi(a,\theta) = 0, \quad \psi_r(a,\theta) = 0 \quad \forall\,\theta\in[0,\pi], \tag{5}$$

and by the asymptotic condition

$$\psi(r,\theta) = \tfrac{1}{2}r^2(\sin\theta)^2\,U + o(r^2) \quad \text{as} \quad r\to\infty, \tag{6}$$

which represents the prescribed uniform flow at infinity.

Now (6) shows the sign of ψ to change with the sign of U, and the homogeneous boundary conditions (5) are unaffected by such a change. Being quadratic in ψ, the right-hand side of (4) is invariant under the transformation $\psi\to-\psi$, whereas the linear left-hand side has its sign changed. But the whole problem is invariant under the composite transformation $U\to-U$, $\psi\to-\psi$, $\theta\to\pi-\theta$, because

$$\partial/\partial(\pi-\theta) = -\partial/\partial\theta, \quad \sin(\pi-\theta) = \sin\theta, \quad \cot(\pi-\theta) = -\cot\theta$$

and the second-order operator L is invariant. (This discrete symmetry is to be expected, of course, because the change $\theta\to\pi-\theta$ amounts to reversing the direction of the axis of the polar coordinate system.) Accordingly, if the problem is recast in dimensionless form as is standard, with r now connoting r/a and ψ now connoting ψ/a^2U which is independent of the sign of U, equation (4) becomes

$$\mathrm{L}^2\psi = \frac{R}{r^2\sin\theta}\left(\psi_\theta\frac{\partial}{\partial r} - \psi_r\frac{\partial}{\partial\theta} + 2\psi_r\cot\theta - \frac{2}{r}\psi_\theta\right)\mathrm{L}\psi, \tag{7}$$

in which $R = aU/\nu$ must be allowed to change sign with U in order to accommodate the transformation $\theta\to\pi-\theta$ that we have just noted to be concomitant with $U\to-U$. Thus the needed solution of (7) satisfying

$$\psi(1,\theta) = 0, \quad \psi_r(1,\theta) = 0 \quad \forall\,\theta\in[0,\pi] \tag{8}$$

and

$$\psi(r,\theta) = \tfrac{1}{2}r^2\sin^2\theta + o(r^2) \quad \text{as} \quad r\to\infty \tag{9}$$

is the same in either case except that $\psi(r,\theta)$ when $R > 0$ becomes $\psi(r,\bar\theta)$ with $\bar\theta = \pi-\theta$ when $R < 0$.

Correspondingly, the dimensionless velocity components $u(r,\theta)$ and $v(r,\theta)$ when $R > 0$ become $-u(r,\bar\theta)$ and $-v(r,\bar\theta)$ when $R < 0$, if v is redefined to be measured in the direction of increasing $\bar\theta$. It follows then that

$$(u,v)\,(r,\bar\theta) = (r\sin\bar\theta)^{-1}\left(r^{-1}\frac{\partial}{\partial\bar\theta}, -\frac{\partial}{\partial r}\right)\psi(r,\bar\theta),$$

and so the dimensionless velocities are exactly the same as in the case $U > 0$. All these considerations are more or less obvious from the symmetry of the hydrodynamic problem, but it is helpful to identify the details.

Calculation of the drag involves three steps. First, a solution of (7) is found satisfying (8) and (9). Then the pressure p in the fluid is found from either of the Navier–Stokes equations in spherical polar coordinates (see Batchelor 1967, p. 601), which respectively relate the components p_r and $r^{-1}p_\theta$ of ∇p to the velocity (u,v) and hence to ψ. Either of these equations confirms the expected symmetry that $p(r,\theta)$ when $R > 0$ becomes $p(r,\bar\theta)$ when $R < 0$. Alternatively, p can be considered to satisfy a form of Poisson's equation given by taking the divergence of the Navier–Stokes system, namely

$$-\Delta p = RQ(\psi),$$

where Δ is the Laplacian operator in spherical polar coordinates and Q is a quadratic operator that is easily seen to have the property $Q\psi(r,\bar{\theta}) = -Q\psi(r,\theta)$. Thus the right-hand side of this equation is invariant as expected to the transformation $R \to -R$, $\theta \to \bar{\theta}$, and Δ is obviously invariant. Note that p is a harmonic function in the Stokes limit $R = 0$.

The remaining step is to evaluate an integral over the surface of the sphere $r = 1$ expressing the net contributions to C_D (the non-dimensional drag D) from shear stress $R^{-1}v_r(1,\theta)$ and normal stress $-p(1,\theta)+2R^{-1}u_r(1,\theta) = -p(1,\theta)$. The integral is found to be

$$C_D = 2\pi \int_0^{\pi} \left\{ -p(1,\theta)\cos\theta + \frac{1}{R}\psi_{rr}(1,\theta) \right\} \sin\theta \, d\theta. \tag{10}$$

This expression shows clearly, as may be expected, that $C_D \to -C_D$ when $R \to -R$ and $\theta \to \bar{\theta} = \pi - \theta$. The first component of the integral changes sign because $\cos\bar{\theta} = -\cos\theta$; the factors p and ψ_{rr} in the integrand are invariant.

Solution of the full equation (7) and exact completion of the other steps needed to find D are inaccessible except by numerical means. An approximate solution of (7) to $O(R)$ for small R was found by Oseen (1910), to be recalled presently, and approximations to higher order in R depend on delicate uses of the method of matched asymptotic expansions. The point being emphasized here is that the simple symmetry of the full problem should be incorporated consistently in such methods. When it is, the non-analytic function $|R|$ of R will inevitably feature in approximations to RC_D, which has to be an even function of $R \in \mathbb{R}$.

The Oseen approximation

Oseen's improvement on the Stokes result $RC_D = 6\pi$ is derived by writing

$$(u, v) = (\cos\theta, -\sin\theta) + (u', v')$$

and linearizing the Navier–Stokes equations in (u', v') (cf. Batchelor 1967, §4.10). Equivalently, writing for the stream function

$$\psi = \tfrac{1}{2}r^2 \sin^2\theta + \psi'$$

and linearizing (7) in ψ', one obtains

$$\mathrm{L}^2\psi' = R\left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\mathrm{L}\psi' \tag{11}$$

in the case $R > 0$; and, in the case $R < 0$, one obtains as the equation for $\psi'(r,\bar{\theta})$

$$\mathrm{L}^2\psi' = |R|\left(\cos\bar{\theta} \frac{\partial}{\partial r} - \frac{\sin\bar{\theta}}{r}\frac{\partial}{\partial\bar{\theta}}\right)\mathrm{L}\psi'. \tag{12}$$

Oseen found a function satisfying (11) approximately to $O(R)$ which, when added to a simple function f satisfying $\mathrm{L}f = 0$, gives an expression that recovers to this order the solution of the Stokes problem in a neighbourhood $rR \ll 1$ of the sphere and complies with the condition (9). To include the case of negative as well as positive R, the approximate solution of (11) and (12) may be written

$$\psi' = -\tfrac{3}{2}(1-c)\frac{1-\exp\left[-\tfrac{1}{2}|R|(1+c)\,r\right]}{|R|}, \tag{13}$$

where $c = \cos\theta$ when $R > 0$ and $c = \cos\bar{\theta}$ when $R < 0$.

To confirm the approximation, we may note that

$$\mathrm{L}\psi' = \tfrac{3}{2}\sin^2\theta\left(\tfrac{1}{2}|R| + \frac{1}{r}\right)\exp\{-\tfrac{1}{2}|R|(1+c)\,r\}.$$

Hence both sides of (11) and (12) are found to give

$$\tfrac{9}{2}|R|\frac{c\sin^2\theta}{r^2}\exp\{-\tfrac{1}{2}|R|(1+c)\,r\} + O(R^2).$$

The composite approximation used by Oseen is

$$\psi = \tfrac{1}{2}\sin^2\theta\left(r^2 + \frac{1}{2r}\right) + \psi', \tag{14}$$

with ψ' specified by (13). Then for $r = O(1)$ we have

$$\psi = \tfrac{1}{2}\sin^2\theta\left(r^2 + \frac{1}{2r} - \frac{3}{2}r\right) + O(R),$$

which recovers the solution of the Stokes problem and satisfies (8) to $O(1)$ for small $|R|$. And (14) evidently satisfies the asymptotic condition (9).

The use of (14) to evaluate C_D leads to $RC_D = 6\pi(1 + \tfrac{3}{8}|R|)$, which is equivalent to the first two terms of (3). The details are straightforward, virtually reproducing the steps of Oseen's derivation, and so they can be omitted. As both positive and negative small values of R are covered explicitly in the present revised theory, the non-analytic term involving $|R|$ is established without any guesswork, although the outcome is wholly in accord with intuition.

Further comments

Regarding boundary-value problems for elliptic nonlinear equations on *finite* domains, it is well known that integral properties of solutions depend very smoothly – in general analytically – on parameters except at bifurcation points where the number of solutions changes. The familiar but in fact pathological case examined above, in which the plainly meaningful non-dimensional property $D/\mu aU$ is necessarily an even function of $R \in \mathbb{R}$ and so is non-analytic because of the small-R form shown in (1), acquires its peculiar behaviour from the presumed infinite extent of the fluid. It seems well justified to conjecture that, if the relation $D/\mu aU = F(R)$ were known exactly for the motion of the sphere along the axis in a fluid contained by a rigid cylinder of large but finite radius, then the even function $F: \mathbb{R} \to (0, \infty)$ would be analytic, at least for small $|R|$ where the possibility of bifurcations is remote. But an explicit full solution of this harder problem is unlikely ever to be available.

Note that non-analyticity is easily exemplified in the limit as parameters of analytic functions are taken to infinity. For example, the even real function $|R|$ with domain $R \in \mathbb{R}$ arises from the set of analytic even functions $R\tanh\beta R$ ($\beta \in (0, \infty)$) in the limit as $\beta \to \infty$.

The linearized equation (11) or its equivalent has been solved approximately by Goldstein (1929) and others to high powers of R. Estimates of RC_D on this basis, extending Oseen's original approximation to $O(R)$, have been found to compare reasonably well with numerical and experimental results (see Van Dyke 1975, pp. 5, 206, 210, 216), even though they avoid the logarithmic terms that Proudman & Pearson (1957) have shown to arise from a self-consistent theory based on the full Navier–Stokes equations. It has been noted that the convergence of the extended Oseen series can be improved by rearrangement of its terms as quotients, and such reckoning has indicated a singularity at $R = -2.09086\ldots$ (Van Dyke 1970) if the series derived for $R \geqslant 0$ is extrapolated without modification to negative values of R. This spurious singularity can easily be understood by the theory of Padé approximants, and it has nothing whatever to do directly with the physical problem which requires any series expansion for D to extend to negative R in the manner shown by equation (3).

3. Another example of non-analyticity

The mathematical peculiarities associated with an infinite expanse of incompressible fluid are encountered in various other useful applications of this abstraction. The following example, which is perhaps easier to understand in full than the case considered above, will also serve to illustrate the phenomenon of singular behaviour arising in the limit as the boundary of a hydrodynamic model is taken to infinity.

Let us recall the properties of infinitesimal waves in a two-fluid system. A perfect fluid of density ρ_1, bounded above by a horizontal rigid plane, lies at rest in a layer of thickness h above a second perfect fluid of density $\rho_2 > \rho_1$ which lies at rest in a layer of thickness H bounded below by another horizontal rigid plane. This system can be disturbed by wave motions such that the interface between the fluids suffers vertical displacements in the form $\epsilon \sin(\omega t - kx + b)$, where ϵ is an infinitesimal constant, t is time and x horizontal distance, and ω, k and b are arbitrary real numbers. Being supposedly started from rest, the motion in each layer is irrotational; and on this basis the dispersion relation between phase velocity $c = \omega/k$ and wavenumber k is found to be

$$c^2 = \frac{g(\rho_2 - \rho_1)}{k(\rho_1 \coth kh + \rho_2 \coth kH)}, \tag{15}$$

where g is the gravity constant (Lamb 1932, p. 371). Thus $c(k)$, the positive square-root of $c^2(k)$ given by (15), is an even function of k having a unique maximum given by

$$c^2(0) = \frac{g(\rho_2 - \rho_1)}{(\rho_1/h) + (\rho_2/H)} = c_0^2, \tag{16}$$

say, at $k = 0$ (i.e. in the long-wave limit). Moreover, for any given finite values of h and H, the function $c: \mathbb{R} \to (0, c_0)$ is *analytic*, that is, infinitely differentiable and identical with its Taylor-series expansion relative to any point of the real line.

Consider now the result of taking the limit $H \to \infty$ in (15). Because

$$\lim_{H\to\infty} \coth kH = 1 \quad \text{if} \quad k > 0,$$
$$= -1 \quad \text{if} \quad k < 0,$$

(15) gives in the limit

$$c^2 = \frac{g(\rho_2 - \rho_2)}{\rho_1 k \coth kh + \rho_2 |k|} \tag{17}$$

and $c_0^2 = gh\{(\rho_2/\rho_1) - 1\}$. This result shows that the function $c: \mathbb{R} \to (0, c_0)$ is no longer analytic, its first derivative being discontinuous at $k=0$. For long waves, an approximation of the new dispersion relation is

$$c = c_0\left\{1-\frac{1}{2}\frac{\rho_2}{\rho_1}h|k|+\left(\frac{3}{8}\frac{\rho_2}{\rho_1}-\frac{1}{6}\right)h^2k^2+O(|k|^3)\right\}, \tag{18}$$

whereas the notably different long-wave approximation deriving from (15) to (16) is

$$c = c_0\left\{1-\frac{1}{6}\left(\frac{\rho_1 h+\rho_2 H}{\rho_1 H+\rho_2 h}\right)hHk^2+O(k^4)\right\}. \tag{19}$$

The qualitative difference arising in the limit $H\to\infty$, namely the non-analytic expression (17) instead of the analytic expression (15) and the approximate dispersion relation (18) differing from (19), have a profound influence on the general behaviour of this wave system. The practical import of these differences is made more conspicuous when the theory is extended to account for nonlinear as well as dispersive effects, in particular when rational approximations are derived for the interesting and comparatively tractable case of unidirectional long waves. Linked with a first-order account of nonlinear effects, the approximate dispersion relation (19) leads to the Korteweg–de Vries equation. But the alternative relation (18), implying the dispersion of long waves to be dominated by a process represented in the term $O(|k|)$, leads to a radically different evolutionary model, namely the nonlinear pseudo-differential operator equation that is usually called the Benjamin–Ono equation (after Benjamin 1967 and Ono 1975).

The initial-value (Cauchy) problem for the Benjamin–Ono equation presents behaviour that differs in major respects from corresponding behaviour governed by the KdV equation. In particular, asymptotic characterizations of localized initial data are not conserved by solutions in the way known for solutions of the KdV equation. Thus the non-analyticity of basic properties that is induced by the infinite extent of the fluid motion has mathematical consequences eclipsing intuitive appraisal of such wave models. For a careful account of these issues, reference may be made to a recent study by Bona & Saut (1992).

4. Conclusion

The idea presented in §§1 and 2 is simple enough to be immediately acceptable, even without a detailed defence as offered here. It seems incontrovertible, and may perhaps be taken for granted already by many hydrodynamicists. But there are evidently different ways of looking at the matter; as I noted at the start, at least one esteemed colleague has views about it that differ somewhat from mine. He rightly regards the Reynolds number R as a scaling parameter for which an attribution of sign is unnecessary. My standpoint is that a mathematically complete account of the parameterization, allowing R to have either sign, is needed to secure a demonstration that the even function $D/6\pi aU$ of R is non-analytic, which property owes to the infinite extent of the fluid and is a surprising one for a boundary-value problem of elliptic type. The absence of any explicit reference to the matter in the literature, as far as I know, suggests that it now deserves to be spelled out.

I am encouraged by recalling a conversation with Ian Proudman in Cambridge over 30 years ago. He and I were then the first Assistant Editors of the JFM. The idea re-examined here was put to him and, as I remember, he agreed wholeheartedly with it.

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